

On equality of central and class preserving automorphisms of finite p -groups

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Abstract

Let G be a finite non-abelian p -group, where p is a prime. Let $\text{Aut}_c(G)$ and $\text{Aut}_z(G)$ respectively denote the group of all class preserving and central automorphisms of G . We give necessary and sufficient condition for G such that $\text{Aut}_c(G) = \text{Aut}_z(G)$ and classify all finite non-abelian p -groups G with elementary abelian or cyclic center such that $\text{Aut}_c(G) = \text{Aut}_z(G)$. We also characterize all finite p -groups G of order $\leq p^7$ such that $\text{Aut}_c(G) = \text{Aut}_z(G)$ and complete the classification of all finite p -groups of order $\leq p^5$ for which there exist non-inner class preserving automorphisms.

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1 Introduction

Let G be a finite group. Let $\text{Aut}(G)$ and $\text{Inn}(G)$ respectively denote the group of all automorphisms and inner automorphisms of G . An automorphism α of G is called a class preserving automorphism if $\alpha(x) \in x^G$, the conjugacy class of x in G , for all $x \in G$ and an automorphism ϕ of G is called a central automorphism if $g^{-1}\phi(g) \in Z(G)$ for all $g \in G$. The set $\text{Aut}_c(G)$ of all class preserving automorphisms and the set $\text{Aut}_z(G)$ of all central automorphisms of G are normal subgroups of $\text{Aut}(G)$. Let $\text{Aut}_z^z(G)$ denote the group of all central automorphisms of G fixing the center $Z(G)$ of G element wise. Interest in the equality of class preserving and inner automorphisms dates back to 1911 when Burnside [4, p. 463] posed the following question: Does there exist a finite group G such that G has a non-inner class preserving automorphism? For details readers can see the excellent survey article by Yadav [19]. In the recent past, interest of many mathematicians turned on the equalities of $\text{Aut}_z(G)$ and $\text{Inn}(G)$, $\text{Aut}_z(G)$ and $Z(\text{Inn}(G))$, $\text{Aut}_z(G)$ and $\text{Aut}_z^z(G)$, and $\text{Aut}_z^z(G)$ and $\text{Inn}(G)$. For example (see [2], [5], [6], [7], [10] and [18]). Curran and McCaughan [6] characterized

finite p -groups G for which $\text{Aut}_z(G) = \text{Inn}(G)$. They proved that if G is a finite p -group, then $\text{Aut}_z(G) = \text{Inn}(G)$ if and only if $\gamma_2(G) = Z(G)$ and $Z(G)$ is cyclic. As a consequence of our results, we give an easy and very short proof of this result in section 3. Yadav [19], in his survey article, has asked to classify all finite p -groups G of class 2 such that $\text{Aut}_c(G) = \text{Aut}_z(G)$. In section 3, we give a necessary and sufficient condition for a finite non-abelian p -group G such that $\text{Aut}_c(G) = \text{Aut}_z(G)$. As far as authors' knowledge is concerned, this is the first necessary and sufficient condition on the equality of $\text{Aut}_c(G)$ with any other group of automorphisms of G . We also classify all finite p -groups G of class 2 for which $\text{Aut}_c(G) = \text{Aut}_z(G)$ in the cases when $Z(G)$ is cyclic or elementary abelian. In section 4, we characterize all finite p -groups G of order $\leq p^7$ for which $\text{Aut}_c(G) = \text{Aut}_z(G)$. We also complete the classification of all finite p -groups G of order $\leq p^5$ for which $\text{Aut}_c(G) \neq \text{Inn}(G)$. It follows from [12] that for extra special p -groups, and hence for all groups of order p^3 , all class preserving automorphisms are inner. From [13], it follows that $\text{Aut}_c(G) = \text{Inn}(G)$ for all groups G of order p^4 . Yadav [17] has found all the groups of order p^5 , p an odd prime, for which $\text{Aut}_c(G) \neq \text{Inn}(G)$. In the present paper we show that out of 51 groups of order 32, there are only two groups for which $\text{Aut}_c(G) \neq \text{Inn}(G)$. In section 5, as an application of our results, we find all finite p -groups G of order p^n ($n \leq 6$) for which $\text{Aut}_c(G) = \text{Aut}_z(G)$ from the lists of all such p -groups given by James [11] (for odd p) and Hall and Senior [8] (for $p = 2$).

2 Notations and Preliminaries

By $\text{Hom}(G, A)$ we denote the group of all homomorphisms of G into an abelian group A , by $d(G)$ we denote the rank of G , and by C_{p^n} we denote the cyclic group of order p^n . A non-abelian group G that has no non-trivial abelian direct factor is said to be purely non-abelian. Observe that a group G is purely non-abelian if its center $Z(G)$ is contained in the frattini subgroup $\Phi(G)$ of G . For two subgroups H and K of G , $[H, K]$ denotes the subgroup of G generated by all commutators $[x, y] = x^{-1}y^{-1}xy$ with $x \in H$ and $y \in K$. By $[x, G]$ we denote the set of all commutators of the form $[x, g]$, $g \in G$. Observe that if G is nilpotent of class 2, then $[x, G]$ is a normal subgroup of G . The lower central series of a group G is the descending series $G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_i(G) \geq \cdots$, where $\gamma_{n+1}(G) = [\gamma_n(G), G]$, and upper central series is the ascending series $Z(G) = Z_1(G) \leq Z_2(G) \leq \cdots \leq Z_i(G) \leq \cdots$, where $Z_{i+1}(G) = \{x \in G \mid [x, y] \in Z_i(G) \text{ for all } y \in G\}$. Let G be a finite p -group and M be non-trivial proper normal subgroup of G . The pair (G, M) is called a Camina pair if $M \subseteq [y, G]$ for all $y \in G - M$ and G is called a Camina group if $(G, \gamma_2(G))$ is a Camina pair.

The following well known result of Adney and Yen [1] will be referred to as Adney-Yen Lemma.

Lemma 2.1 *If G is a purely non-abelian group, then there is a one-to-one correspondence between $\text{Aut}_z(G)$ and $\text{Hom}(G/\gamma_2(G), Z(G))$.*

And the following result of Morigi [15, Lemma 0.4] will be referred to as Morigi Lemma.

Lemma 2.2 *Let G be a finite nilpotent group of class 2. Then $\exp(\gamma_2(G)) = \exp(G/Z(G))$ and in the decomposition of $G/Z(G)$ in direct product of cyclic groups at least two factors of maximal order must occur.*

The well known commutator identities

$$[x, yz] = [x, z][x, y][x, y, z] ; \quad [xy, z] = [x, z][x, z, y][y, z],$$

where $x, y, z \in G$, will be frequently used without any reference.

3 Main Results

We start with the following theorem which gives a necessary and sufficient condition on a finite p -group G such that $\text{Aut}_c(G) = \text{Aut}_z(G)$.

Theorem 3.1 *Let G be a finite p -group. Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $\text{Aut}_c(G) \approx \text{Hom}(G/Z(G), \gamma_2(G))$ and $\gamma_2(G) = Z(G)$.*

Proof. First assume that $\text{Aut}_c(G) = \text{Aut}_z(G)$. We show that $Z(G) \leq \Phi(G)$. Suppose there exists a maximal subgroup M of G such that $Z(G)$ is not contained in M . Let $h \in Z(G) - M$, then $G = M \langle h \rangle$. There exists a non-trivial element z in $Z(G) \cap \Phi(G)$ of order p . It is easy to check that the map $\mu : G \rightarrow G$ defined by $\mu(mh^i) = mh^i z^i$ for every $m \in M$ and for every i , $0 \leq i \leq p-1$, is a central automorphism of G which is not class preserving and thus $Z(G) \leq \Phi(G)$. For any commutator $[a, b] \in G$, where $a, b \in G$, we can define an inner automorphism $\iota_b : G \rightarrow G$ such that $a^{-1}\iota_b(a) = [a, b] \in Z(G)$. Thus $\gamma_2(G) \leq Z(G)$. For any $\mu \in \text{Aut}_c(G)$, the map $\psi_\mu : G/Z(G) \rightarrow \gamma_2(G)$ defined as $b\psi_\mu(bZ(G)) = \mu(b)$ is a homomorphism. It is easy to see that the map ψ sending μ to ψ_μ is a monomorphism of the group $\text{Aut}_c(G)$ into the group $\text{Hom}(G/Z(G), \gamma_2(G))$. For any $\tau \in \text{Hom}(G/Z(G), \gamma_2(G))$, the map $\mu : G \rightarrow G$ defined as $\mu(g) = g\tau(gZ(G))$, where $g \in G$, is a central automorphism and $\psi(\mu) = \psi_\mu = \tau$. Thus ψ is onto as well and hence $\text{Aut}_c(G) \approx \text{Hom}(G/Z(G), \gamma_2(G))$. But, by Adney-Yen Lemma, $|\text{Aut}_z(G)| = |\text{Aut}_c(G)| = |\text{Hom}(G/\gamma_2(G), Z(G))|$, and so

$$|\text{Hom}(G/Z(G), \gamma_2(G))| = |\text{Hom}(G/\gamma_2(G), Z(G))|. \quad (1)$$

Suppose to the contrary that $\gamma_2(G) < Z(G)$. Then $G/Z(G)$ is a proper quotient of $G/\gamma_2(G)$ and $|(G/\gamma_2(G))/(G/Z(G))| = |Z(G)/\gamma_2(G)| > 1$. It thus follows from [5, Lemma 2.8] that $\text{Hom}(G/Z(G), \gamma_2(G))$ is isomorphic to a proper subgroup of $\text{Hom}(G/\gamma_2(G), Z(G))$. This is a contradiction to (1) and hence $\gamma_2(G) = Z(G)$.

Conversely assume that $\text{Aut}_c(G) \approx \text{Hom}(G/Z(G), \gamma_2(G))$ and $\gamma_2(G) = Z(G)$. Observe that since $\gamma_2(G) = Z(G)$, $\text{Aut}_c(G) \leq \text{Aut}_z(G)$ and G is purely non-abelian. By Adney-Yen Lemma

$$|\text{Aut}_z(G)| = |\text{Hom}(G/\gamma_2(G), Z(G))| = |\text{Hom}(G/Z(G), \gamma_2(G))| = |\text{Aut}_c(G)|.$$

This completes the proof. \square

As a consequence of Theorem 3.1, we give the following easy proof of main result of Curran and McCaughan [6]. The proof of only if part is easy and is similar to as given in [6]. But we give it here for the sake of completeness.

Theorem 3.2 *If G is a finite p -group, then $\text{Aut}_z(G) = \text{Inn}(G)$ if and only if $\gamma_2(G) = Z(G)$ and $Z(G)$ is cyclic.*

Proof. Suppose $\gamma_2(G) = Z(G)$ and $Z(G)$ is cyclic. Then $\text{Inn}(G) \leq \text{Aut}_z(G)$, $\exp(G/Z(G)) = \exp(\gamma_2(G))$ by Morigi Lemma, and G is purely non-abelian. By Adney-Yen lemma

$$|\text{Aut}_z(G)| = |\text{Hom}(G/\gamma_2(G), Z(G))| = |G/Z(G)| = |\text{Inn}(G)|,$$

and thus $\text{Aut}_z(G) = \text{Inn}(G)$.

Conversely suppose that $\text{Inn}(G) = \text{Aut}_z(G)$. Then, as in above theorem, we can prove that $\text{Aut}_c(G) \approx \text{Hom}(G/Z(G), \gamma_2(G))$ and $\gamma_2(G) = Z(G)$. Thus nilpotency class of G is 2 and hence $\exp(G/Z(G)) = \exp(\gamma_2(G))$. It now follows from

$$\text{Hom}(G/Z(G), \gamma_2(G)) \approx \text{Aut}_z(G) = \text{Inn}(G) \approx G/Z(G),$$

that $\gamma_2(G)$ is cyclic. \square

Theorem 3.3 *Let G be a finite non-abelian p -group such that $Z(G)$ is elementary abelian. Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if G is a Camina p -group of nilpotency class 2.*

Proof. Suppose first that G is a Camina p -group of nilpotency class 2. Then $\gamma_2(G) = Z(G)$ by [14, Lemma 2.1]. Let $\phi \in \text{Aut}_c(G)$ and $g \in G$. Then $g^{-1}\phi(g) \in \gamma_2(G) = Z(G)$. Thus ϕ is a central automorphism. On the other hand, let $\psi \in \text{Aut}_z(G)$ and let $h \in G$. If $h \in Z(G)$, then $\psi(h) = h$ and if $h \in G - Z(G)$, then $h^{-1}\psi(h) \in Z(G) \subseteq [h, G]$. It thus follows that $\psi(h) = b^{-1}hb$ for some $b \in G$. Therefore ψ is a class preserving automorphism and hence $\text{Aut}_c(G) = \text{Aut}_z(G)$.

Conversely suppose that $\text{Aut}_c(G) = \text{Aut}_z(G)$. By Theorem 3.1, $\gamma_2(G) = Z(G)$ and $\text{Aut}_c(G) \approx \text{Hom}(G/Z(G), \gamma_2(G))$. It follows that $\exp(G/Z(G)) = \exp(\gamma_2(G)) = \exp(Z(G)) = p$. Let $|G| = p^l$, $\{y_1, y_2, \dots, y_r\}$ be a minimal generating set of G and let $|\gamma_2(G)| = p^s$. Then

$$|\text{Aut}_c(G)| = |\text{Hom}(G/Z(G), \gamma_2(G))| = p^{s(l-s)} = p^{sr}.$$

Any element $y \in G - \gamma_2(G)$ is a part of a minimal generating set $\{y = y_1, y_2, \dots, y_r\}$ for G . If possible suppose $[y, G] < \gamma_2(G)$, then $|y^G| = |[y, G]| < |\gamma_2(G)| = p^s$. Thus $|\text{Aut}_c(G)| < p^{sr}$, a contradiction and hence G is a Camina p -group of class 2. \square

Theorem 3.4 *Let G be a finite non-abelian p -group such that $Z(G)$ is cyclic. Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $Z(G) = \gamma_2(G)$.*

Proof. If $Z(G) = \gamma_2(G)$, then $\text{Inn}(G) = \text{Aut}_z(G)$ by Theorem 3.2. Let μ be any class preserving automorphism of G and let $g \in G$. Then $g^{-1}\mu(g) \in \gamma_2(G) = Z(G)$. Thus μ is a central automorphism and hence

$$\text{Aut}_c(G) \leq \text{Aut}_z(G) = \text{Inn}(G) \leq \text{Aut}_c(G).$$

Converse follows from Theorem 3.1. \square

4 Groups of order p^n , $n \leq 7$

In this section we characterize all finite p -groups of order p^n ($n \leq 7$) such that $\text{Aut}_c(G) = \text{Aut}_z(G)$. For this we need the following theorem which also completes the classification of all finite p -groups of order $\leq p^5$ such that $\text{Aut}_c(G) \neq \text{Inn}(G)$. In the theorem we show that out of 51 groups of order 32, there are only 2 groups for which $\text{Aut}_c(G) \neq \text{Inn}(G)$. A list of groups of order 32 is available from Hall and Senior [8] and from Sag and Wamsley [16] with minimal presentations of groups. In [8], groups of order 32 are divided into 8 isoclinism families. We denote the i th family by Φ_i . As mentioned in [16], the groups are listed in same order in both [8] and [16] and so we take the liberty of choosing the presentation of a group from any of the lists. Unless or otherwise stated, we use Sag and Wamsley's list and adopt the same notations for the nomenclature and presentations of the groups. However, we write the i -th group of order 32 in the list as G_i and generators 1,2,3 and 4 respectively as x, y, z and w . Yadav [17] has proved that if G and H are two finite non-abelian isoclinic groups, then $\text{Aut}_c(G) \approx \text{Aut}_c(H)$. Therefore it is sufficient to pick only one group from each isoclinism family. We need the following lemma to prove the theorem.

Lemma 4.1 *If G is a group of order p^n , $n \geq 3$ and $|Z(G)| = p^{n-2}$, then $\text{Aut}_c(G) = \text{Inn}(G)$.*

Proof. Observe that nilpotency class of G is 2. If x is a non-central element of G , then $|C_G(x)| = p^{n-1}$ and thus $|x^G| = |[x, G]| = p$. If x is a central element of G , then $|x^G| = |[x, G]| = 1$. In any case, $[x, G]$ is cyclic and therefore $\text{Aut}_c(G) = \text{Inn}(G)$ by [17, Theorem 3.5]. \square

Theorem 4.2 *For all groups G of order 32, except for the groups G_{44} and G_{45} in the sixth family Φ_6 , $\text{Aut}_c(G) = \text{Inn}(G)$.*

Proof. The first family Φ_1 contains abelian groups G_1 to G_7 and therefore all class preserving automorphisms for these groups are inner. From the second family Φ_2 , we pick $G_{11} = \{x^2, y^4, z^2, [x, y, x], [x, y, y], [x, z], [y, z]\}$. Clearly z is in the center and since $[x, y^2] = [x, y]^2 = [x^2, y] = 1$, it follows that y^2 is in the center and $|[x, y]| = 2$. Also, since z commutes with x and y , it commutes with $[x, y]$ as well. Thus $[x, y]$ is also in the center and hence $|Z(G_{11})| = 8$. Therefore $\text{Aut}_c(G_{11}) = \text{Inn}(G_{11})$ by above lemma.

From third isoclinism family we take $G_{23} = \{x^8, y^2, z^2, [x, y]x^2, [x, z], [y, z]\}$. Observe that $G_{23} = H \oplus K$, where $H = \{x, y | x^8 = y^2 = 1, yx = x^7y\}$ and $K = \{z | z^2 = 1\}$. It is easily seen that K is cyclic and therefore $\text{Aut}_c(K) = \text{Inn}(K)$. Also, every element of H can be written as $x^i y^j$ for suitable i and j , therefore $\text{Aut}_c(H) = \text{Inn}(H)$ by [13, Proposition 4.1]. Hence $\text{Aut}_c(G_{23}) = \text{Inn}(G_{23})$ by [12, Proposition 2.2].

In Φ_4 , consider $G_{34} = \{x^4, y^4, z^2, [x, y], [x, z]x^2, [y, z]y^2\}$. Let $H_{34} = \langle x, y \rangle$. Then H_{34} is an abelian normal subgroup of order 16 and therefore G_{34}/H_{34} is a cyclic group of order 2. Hence $\text{Aut}_c(G_{34}) = \text{Inn}(G_{34})$ by [9, Proposition 2.7].

From fifth family, consider

$$G_{43} = \{w^2, x^2y^{-2}, x^2z^{-2}, [z, y]x^2, [w, x]x^2, [x, y], [x, z], [y, w], [z, w]\}.$$

Since $d(G_{43}) = 4$, $|\Phi(G_{43})| = |\gamma_2(G_{43})| = 2$. Also, since G_{43} is a stem group, $Z(G_{43}) \leq \gamma_2(G_{43})$. Thus G_{43} is an extra-special group and hence $\text{Aut}_c(G_{43}) = \text{Inn}(G_{43})$ by [12, Theorem 3.2].

Next consider the group $G = G_{44} = \{x^2, z^2, [y, x]y^4, [y, z]y^2, [x, z]\}$ from Φ_6 . Since $yx = xy[y, x] = xyy^{-4} = xy^{-3}$ and since $[y, z] = y^{-2}$ implies that $zy = y^{-1}z$, it follows that every element of G can be written in the form $x^i y^j z^k$, where $0 \leq i, k \leq 1$, and $0 \leq j \leq 7$ and hence $|y| = 8$. Since G is of nilpotency class 3 and $d(G) = 3$, it follows that $|\Phi(G)| = |\gamma_2(G)| = 4$. But $y^2 = [z, y] \in \gamma_2(G)$ is of order 4, therefore $\gamma_2(G) = \langle y^2 \rangle$. Now G_{44} is a stem group, therefore $Z(G) < \gamma_2(G)$. Since $[z, y^4] = [z, y^2]^2 = [z, y]^4 = y^8 = 1$ and $[x, y^2] = [x, y]^2 = y^8 = 1$, $y^4 \in Z(G)$ and thus $Z(G) = \langle y^4 \rangle$. Therefore $G/Z(G)$ is a class 2 group of order 16. Then $Z(G/Z(G))$ is of order 4 and hence $|Z_2(G)| = 8$. We prove that $Z(G) \subseteq [g, G]$ for all $g \in G - \gamma_2(G)$. Let $g = x^i y^j z^k \in G - \gamma_2(G)$. First suppose that j is even. Then $g = y^j x^i z^k$. Both i and k cannot be zero, because then $g \in \gamma_2(G)$. If $k = 1$, then $[g, y^2] = [y^j x^i z, y^2] = [x^i z, y^2] = [z, y^2] = y^4$. If $k = 0$, then $[g, y] = [y^j x, y] = [x, y] = y^4$. Thus $Z(G) \subseteq [g, G]$ for all $g \in G - \gamma_2(G)$ in this case. Next suppose that j is odd. Then $g = y^{j-1} x^i y z^k$. If $k = 1$, then $[g, y^2] = [y^{j-1} (x^i y z), y^2] = [x^i (y z), y^2] = [y z, y^2] = [z, y^2] = [z, y]^2 = y^4$. Suppose $k = 0$ and $i = 1$. Then $[g, y] = [y^{j-1} (xy), y] = [xy, y] = [x, y] = y^4$. Finally suppose $k = 0 = i$. Then $[g, x] = [y^{j-1} y, x] = [y, x] = y^4$. Thus $Z(G) \subseteq [g, G]$ for all $g \in G - \gamma_2(G)$ in this case as well. Therefore, by [17, Lemma 2.2]

$$|\text{Aut}_c(G)| \geq |\text{Aut}_z(G)| |G/Z_2(G)|.$$

Since $Z(G) < \gamma_2(G)$, G is purely non abelian and therefore by Adney-Yen Lemma, we have

$$|\text{Aut}_z(G)| = |\text{Hom}(G/\gamma_2(G), Z(G))| = |\text{Hom}(C_2 \times C_2 \times C_2, C_2)| = 8.$$

Thus $|\text{Aut}_c(G)| \geq 2^5 > 2^4 = |\text{Inn}(G)|$.

From seventh family we take $G_{46} = \{x^2, y^4, [x, y, x], [x, y, y, y]\}$. Let $u = [x, y]$ and $v = [x, y, y] = [u, y]$, then u commutes with x and v commutes with y . Observe that $1 = [x^2, y] = [x, y]^2 = u^2$; $[x, y^2] = [x, y]^2[x, y, y] = v$; $1 = [x, y^4] = [x, y^2]^2[x, y^2, y^2] = v^2$. Thus $|u| = |v| = 2$. Since $xy^2 = (xy)y = (yxu)y = yx(uy) = yx(yuv) = y(xy)uv = y(yxu)uv = y^2xv$, $vx = (xy^2xy^2)x = (xy^2x)y^2x = (xy^2x)(xy^2v) = xv$. Thus v commutes with x and hence with u as well. It is easy to see that every element of G_{46} can be written in the form $x^i y^j u^k v^l$, where $0 \leq i, k, l \leq 1$ and $0 \leq j \leq 3$. Take $H_{46} = \langle x, u, v \rangle$, then H_{46} is an abelian normal subgroup of order 8 such that $G_{46}/H_{46} = \langle yH_{46} \rangle$ is a cyclic group of order 4. Thus $\text{Aut}_c(G_{46}) = \text{Inn}(G_{46})$ by [9, Proposition 2.7].

Finally consider $G_{51} = \{x^8 y^{-2}, [y, x^7]x^2\}$ from Φ_8 . The relation $[x^7, y] = x^2$ implies that $yx^9 = x^7 y$. Post multiplying by $x^8 = y^2$ we have $yx = x^7 y^3 = x^{15} y$. Thus every element of G_{51} can be written as $x^i y^j$, $0 \leq i \leq 15, 0 \leq j \leq 1$ and hence $\text{Aut}_c(G_{51}) = \text{Inn}(G_{51})$ by [13, Proposition 4.1]. \square

Remark: The order of $\text{Aut}_c(G)$ for $G = G_{44}$ in the above theorem is in fact exactly equal to 32. Observe that $C_G(x) = \langle x, y^2, z \rangle$, $C_G(y) = \langle y \rangle$ and $C_G(z) = \langle x, y^4, z \rangle$. Thus $|C_G(x)| = 16$ and $|C_G(y)| = |C_G(z)| = 8$. Hence $|x^G| = 2$ and $|y^G| = |z^G| = 4$. Since any class preserving automorphism preserves the conjugacy classes, there are $|x^G|$, $|y^G|$ and $|z^G|$ choices for the images of x , y and z respectively under it. Thus $|\text{Aut}_c(G)| \leq |x^G||y^G||z^G| = 32$.

Theorem 4.3 *Let G be a non-abelian group of order p^n , $3 \leq n \leq 5$. Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $\gamma_2(G) = Z(G)$ and $Z(G)$ is cyclic.*

Proof. Suppose $\text{Aut}_c(G) = \text{Aut}_z(G)$. Then $\gamma_2(G) = Z(G)$ by Theorem 3.1. If $|G| = p^3$ or p^4 , then $\text{Aut}_c(G) = \text{Inn}(G)$ by [12, 13]. If $|G| = p^5$, then since class of G is 2, it follows from Theorem 4.2 (for $p = 2$) and [17, Theorem 5.5] (for odd p) that $\text{Aut}_c(G) = \text{Inn}(G)$. Thus $\text{Inn}(G) = \text{Aut}_z(G)$ and hence $\gamma_2(G)$ is cyclic by Theorem 3.2. The converse follows from Theorem 3.4. \square

Theorem 4.4 *Let G be a non-abelian group of order p^6 . Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if either $\gamma_2(G) = Z(G)$ and $Z(G)$ is cyclic or G is a Camina p -group of nilpotency class 2.*

Proof. Suppose $\text{Aut}_c(G) = \text{Aut}_z(G)$. Then $\gamma_2(G) = Z(G)$ by Theorem 3.1. If $Z(G)$ is not cyclic, then we prove that $Z(G)$ is elementary abelian and the result will then follow from Theorem 3.3. Now $p^2 \leq |Z(G)| \leq p^4$. If $|Z(G)| = p^2$, then it is trivially elementary abelian. Let $|Z(G)| = p^3$ and let $Z(G) \approx C_{p^2} \times C_p$. Then $\exp(Z(G)) = \exp(\gamma_2(G)) = \exp(G/Z(G)) = p^2$ and hence $G/Z(G) \approx C_{p^2} \times C_p$, a contradiction to Morigi Lemma. If $|Z(G)| = p^4$, then $G/Z(G)$ is

elementary abelian. Therefore $\exp(Z(G)) = \exp(\gamma_2(G)) = \exp(G/Z(G)) = p$ and hence $Z(G)$ is elementary abelian. The converse follows from Theorems 3.3 and 3.4. \square

Theorem 4.5 *Let G be a non-abelian group of order p^7 . Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if either $\gamma_2(G) = Z(G)$ and $Z(G)$ is cyclic or G is a Camina p -group of nilpotency class 2.*

Proof. Suppose $\text{Aut}_c(G) = \text{Aut}_z(G)$. Then $\gamma_2(G) = Z(G)$ by Theorem 3.1. If $Z(G)$ is not cyclic, then as in Theorem 4.4, we need only prove that $Z(G)$ is elementary abelian. Now $p^2 \leq |Z(G)| \leq p^5$. The cases $|Z(G)| = p^2$ or p^5 can be handled as in the above theorem. Let $|Z(G)| = p^3$ and let $Z(G) \approx C_{p^2} \times C_p$, then $G/Z(G) \approx C_{p^2} \times C_{p^2}$ by Morigi Lemma. Thus G is a 2-generator class 2 group and hence $\gamma_2(G)$ is cyclic by [3, Lemma 36.5]. This is a contradiction to $\gamma_2(G) = Z(G) \approx C_{p^2} \times C_p$. Let $|Z(G)| = p^4$ and let $\exp(Z(G)) = p^2$ or p^3 . Then $|G/Z(G)| = p^3$ and $\exp(G/Z(G)) = p^2$ or p^3 , which is not possible by Morigi Lemma. The converse follows from Theorems 3.3 and 3.4. \square

5 Application

In this section, we use the classification of all groups of order p^n , $5 \leq n \leq 6$, given by James [11] for odd p and Hall and Senior [8] for $p = 2$. As an application of our results, we find those groups G of order p^5 and p^6 for which $\text{Aut}_c(G) = \text{Aut}_z(G)$.

Theorem 5.1 *Let G be a non-abelian group of order p^5 , where p is an odd prime. Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $G \in \Phi_5$.*

Proof. By Theorem 4.3, $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $\gamma_2(G) = Z(G)$ and $Z(G)$ is cyclic. This happens if and only if $G \in \Phi_5$ by §4.1 of [11]. \square

Lemma 5.2 *A non-abelian group G of order p^6 , where p is an odd prime, is a Camina group of class 2 if and only if $G \in \Phi_{15}$.*

Proof. In a Camina p -group of class 2, $\gamma_2(G) = Z(G)$ and $|x^G| = |\gamma_2(G)|$ for each $x \in G - Z(G)$. This happens if and only if $G \in \Phi_{15}$ by §4.1 of [11]. \square

Theorem 5.3 *Let G be a non-abelian group of order p^6 , where p is an odd prime. Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $G \in \Phi_{14}$ or Φ_{15} .*

Proof. By Theorem 4.4, $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if either G is a Camina p -group of nilpotency class 2 or $\gamma_2(G) = Z(G)$ and $Z(G)$ is cyclic. Now G is a Camina p -group of class 2 if and only if $G \in \Phi_{15}$ by above lemma, and $\gamma_2(G) = Z(G)$ and $Z(G)$ is cyclic if and only if $G \in \Phi_{14}$ by §4.1 of [11]. \square

Theorem 5.4 *Let G be a non-abelian group of order 2^5 . Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $G \in \Phi_5$.*

Proof. By Theorem 4.3, $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $\gamma_2(G) = Z(G)$ and $Z(G)$ is cyclic. There are only two families *viz.* Φ_4 and Φ_5 which consist of groups G such that $\gamma_2(G) = Z(G)$. Consider the group $H = G_{34} = \{x^4, y^4, z^2, [x, y], [x, z]x^2, [y, z]y^2\}$ from Φ_4 . Observe that $[x^2, z] = [x, z]^2 = x^4 = 1$ and $[y^2, z] = [y, z]^2 = y^4 = 1$. Thus $x^2, y^2 \in Z(H) = \gamma_2(H)$ and hence $|\gamma_2(H)| \geq 4$. Since $d(H) = 3$, $|\Phi(H)| = 4$ and thus $Z(H) = \gamma_2(H) = \Phi(H)$. Now $\exp(H/Z(H)) = \exp(\gamma_2(H))$ implies that $\gamma_2(H)$ is elementary abelian. As proved in Theorem 4.2, the group G_{43} of fifth family Φ_5 is an extra-special group. Since $\gamma_2(G)$ is family invariant, all the groups in fourth family have elementary abelian center and all the groups in fifth family have cyclic center. Hence $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $G \in \Phi_5$. \square

In the next lemma, all the presentations for groups are taken from [8]. For the sake of brevity, we only write the relators. And for the same reason, a relator of the form $[\alpha_i, \alpha_j]$ is excluded from the presentation if α_i commutes with α_j . For the group $G_{(i,j)}$, i stands for the group number and j stands for the family number.

Lemma 5.5 *A non-abelian group G of order 2^6 is a Camina group of class 2 if and only if $G \in \Phi_{13}$.*

Proof. Since $\gamma_2(G) = Z(G)$ for a Camina p -group of class 2, we need only consider the families Φ_9 to Φ_{13} . These families contain stem groups and the structure of conjugacy classes for stem groups is an invariant of the isoclinism family. Therefore it is sufficient to pick one group from each of these families. First consider

$$G = G_{(144,9)} = \langle \alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2, \alpha_5^2, \alpha_6^2, [\alpha_4, \alpha_5]\alpha_1^{-1}, [\alpha_4, \alpha_6]\alpha_2^{-1}, [\alpha_5, \alpha_6]\alpha_3^{-1} \rangle.$$

Since $\alpha_1, \alpha_2, \alpha_3 \in Z(G)$, $|Z(G)| = |\gamma_2(G)| \geq 8$. Therefore if $x \in G - Z(G)$, then $|x^G| = |[x, G]| \leq 4$ and hence G is not a Camina group. Next consider

$$G = G_{(154,10)} = \langle \alpha_1^2, \alpha_2^2, \alpha_4^2, \alpha_6^2, \alpha_3^2\alpha_1^{-1}, \alpha_5^2\alpha_2^{-1}, [\alpha_3, \alpha_4]\alpha_1^{-1}, [\alpha_5, \alpha_6]\alpha_2^{-1} \rangle.$$

Here $\alpha_1, \alpha_2 \in Z(G)$, therefore $|Z(G)| = |\gamma_2(G)| \geq 4$. Since $\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6 \in C_G(\alpha_4)$ and every element of subgroup generated by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6\}$ can be written as $\alpha_1^i \alpha_2^j \alpha_3^k \alpha_5^l \alpha_6^m$, $0 \leq i, j, k, l, m \leq 1$, it follows that $|C_G(\alpha_4)| = 32$. Thus $|\alpha_3^G| = |[\alpha_3, G]| = 2 \neq |\gamma_2(G)|$ and hence G is not a Camina group. Next consider

$$G_{(169,11)} = \langle \alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2, \alpha_6^2, \alpha_5^2\alpha_2^{-1}, [\alpha_4, \alpha_5]\alpha_1^{-1}, [\alpha_3, \alpha_6]\alpha_1^{-1}, [\alpha_5, \alpha_6]\alpha_2^{-1} \rangle.$$

As above, we can show that $|\alpha_3^G| = |[\alpha_3, G]| = 2 \neq |\gamma_2(G)| \geq 4$ and hence $G_{(169,11)}$ is also not a Camina group. We next consider $G = G_{(180,12)}$

$$\langle \alpha_1^2, \alpha_2^2\alpha_1^{-1}, \alpha_3^2\alpha_1^{-1}, \alpha_4^2, \alpha_5^2\alpha_3^{-1}, \alpha_6^2\alpha_4^{-1}, [\alpha_4, \alpha_5]\alpha_1^{-1}, [\alpha_3, \alpha_6]\alpha_1^{-1}, [\alpha_5, \alpha_6]\alpha_2^{-1} \rangle.$$

If $|Z(G)| = 16$, then $G/Z(G)$ and hence $\gamma_2(G)$ is elementary abelian which is not so because $\alpha_2 = [\alpha_5, \alpha_6] \in \gamma_2(G)$ is of order 4. If $|Z(G)| = 8$, then

$Z(G) \approx C_2^2 \times C_2$ or $Z(G) \approx C_2 \times C_2 \times C_2$. The first case is not possible by Morigi Lemma and the second one is ruled out because $\alpha_2 = [\alpha_5, \alpha_6] \in \gamma_2(G) = Z(G)$ is of order 4. Thus $Z(G)$ is a cyclic group, generated by α_2 , of order 4. Therefore G cannot be a Camina group by [14, Lemma 2.1, Theorem 2.2]. Finally consider

$$G = G_{(183,13)} = \langle \alpha_1^2, \alpha_2^2, \alpha_3^2 \alpha_1^{-1}, \alpha_4^2 \alpha_2^{-1}, \alpha_5^2, \alpha_6^2, [\alpha_3, \alpha_5] \alpha_1^{-1}, \\ [\alpha_4, \alpha_5] \alpha_2^{-1} \alpha_1^{-1}, [\alpha_3, \alpha_6] \alpha_2^{-1} \alpha_1^{-1}, [\alpha_4, \alpha_6] \alpha_2^{-1} \rangle.$$

Observe that $\gamma_2(G) = Z(G) = \langle \alpha_1, \alpha_2 \rangle$ and every element g of G is of the form $z \alpha_3^{m_3} \alpha_4^{m_4} \alpha_5^{m_5} \alpha_6^{m_6}$, where $z \in Z(G)$ and $0 \leq m_3, m_4, m_5, m_6 \leq 1$. We claim that $Z(G) \subseteq [g, G]$ for all $g \in G - Z(G)$. First suppose that $m_6 = 1$. If $m_5 = 0$, then $[g, \alpha_4] = [\alpha_5^{m_5} \alpha_6, \alpha_4] = \alpha_2$ where as $[g, \alpha_3] = \alpha_1 \alpha_2$. And if $m_5 = 1$, then $[g, \alpha_4] = \alpha_1$ where as $[g, \alpha_3] = \alpha_2$. Now suppose that $m_6 = 0$ and $m_5 = 1$. Then $[g, \alpha_3] = \alpha_1$ and $[g, \alpha_4] = \alpha_1 \alpha_2$. We next suppose that $m_5 = m_6 = 0$ and $m_4 = 1$. If $m_3 = 0$, then $[g, \alpha_6] = \alpha_2$ where as $[g, \alpha_5] = \alpha_1 \alpha_2$. And if $m_3 = 1$, then $[g, \alpha_6] = \alpha_1$ where as $[g, \alpha_5] = \alpha_2$. Finally suppose that $m_4 = m_5 = m_6 = 0$ and $m_3 = 1$. Then $[g, \alpha_5] = \alpha_1$ and $[g, \alpha_6] = \alpha_1 \alpha_2$. In every possibility, $Z(G) \subseteq [g, G]$ for all $g \in G - Z(G)$. This proves the claim and the lemma. \square

Theorem 5.6 *Let G be a non-abelian group of order 2^6 . Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $G \in \Phi_{12}$ or Φ_{13} .*

Proof. First suppose that $\text{Aut}_c(G) = \text{Aut}_z(G)$. By Theorem 4.4, either G is a Camina group of class 2 or $\gamma_2(G) = Z(G)$ and $Z(G)$ is cyclic. If G is a Camina group of class 2, then $G \in \Phi_{13}$ by above lemma. Assume $\gamma_2(G) = Z(G)$ and $Z(G)$ is cyclic. There are only five isoclinism families viz. Φ_9 to Φ_{13} which consist of groups G such that $\gamma_2(G) = Z(G)$. Consider $G_{144} \in \Phi_9$. By above lemma $|Z(G_{144})| \geq 8$ and by [16], $d(G_{144}) = 3$. Thus $Z(G_{144}) = \gamma_2(G_{144}) = \Phi(G_{144})$. Now $\exp(G_{144}/Z(G_{144})) = \exp(\gamma_2(G_{144}))$ implies that $Z(G_{144})$ is elementary abelian of order 8. Next consider $G_{154} \in \Phi_{10}$. By above lemma $|Z(G_{154})| \geq 4$ and by [16], $d(G_{154}) = 4$. Thus $Z(G_{154}) = \gamma_2(G_{154}) = \Phi(G_{154})$. Then $\exp(G_{154}/Z(G_{154})) = \exp(\gamma_2(G_{154}))$ implies that $Z(G_{154})$ is elementary abelian of order 4. Next consider $G_{169} \in \Phi_{11}$. By above lemma $|Z(G_{169})| \geq 4$ and by [16], $d(G_{169}) = 4$. Thus $Z(G_{169}) = \gamma_2(G_{169}) = \Phi(G_{169})$. Therefore $\exp(G_{169}/Z(G_{169})) = \exp(\gamma_2(G_{169}))$ implies that $Z(G_{169})$ is elementary abelian of order 4. Finally, if $G \in \Phi_{12}$, then $Z(G)$ is cyclic by Lemma 5.5. \square

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